A family of surfaces with $p_g = q = 2$, $K^2 = 7$ and Albanese map of degree 3

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Abstract

We study a family of surfaces of general type with $p_g = q = 2$ and $K^2 = 7$, originally constructed by Cancian and Frapporti by using the Computer Algebra System `MAGMA`. We provide an alternative, computer-free construction of these surfaces, that allows us to describe their Albanese map and their moduli space.

0   Introduction

In recent years, the work of several authors on the classification of *irregular* algebraic surfaces (that is, surfaces S with $q(S) > 0$) produced a considerable amount of results, see for example the survey papers [**BaCaPi06**, **MP12**] for a detailed bibliography on the subject.

In particular, surfaces of general type with $\chi(\mathcal{O}_S) = 1$, that is, $p_g(S) = q(S)$ were investigated. For these surfaces, [**Deb81**, Théorème 6.1] implies $p_g \leq 4$. Surfaces with $p_g = q = 4$ and $p_g = q = 3$ are nowadays completely classified, see [**Be82**, **CaCiML98**, **HP02**, **Pir02**]. On the other hand, for the the case $p_g = q = 2$, which presents a very rich and subtle geometry, we have so far only a partial understanding of the situation; we refer the reader to [**PePol13a**, **PePol13b**, **PePol14**] for an account on this topic and recent results.

As the title suggest, in this paper we consider a family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$. The existence of such surfaces was originally established in [**CanFr15**] with the help of the Computer Algebra System `MAGMA` (http://magma.maths.usyd.edu.au/magma/); the present work provides an alternative, computer free construction of them, that allows us to describe their Albanese map and their moduli space.

Our results can be summarized as follows, see Theorem 2.7.

Main Theorem. *There exists a 3-dimensional family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ such that, for all elements $S \in \mathcal{M}$, the canonical class K_S is ample and the Albanese map $\alpha \colon S \longrightarrow A$ is a generically finite triple cover of a principally polarized abelian surface (A, Θ), simply branched over a curve D_A numerically equivalent to 4Θ having an ordinary sextuple point and no other singularities. The family $\mathcal{M}$ provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space $\mathcal{M}_{2,2,7}^{\mathrm{can}}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$.*

In particular, this means that $\mathcal{M}$ provides a dense open set of a generically smooth, irreducible component of $\mathcal{M}_{2,2,7}^{\mathrm{can}}$.

Let us explain now how the paper is organized. In Section 1 we explain our construction in detail and we compute the invariants of the resulting surfaces (Proposition 1.5); moreover we study their Albanese map, giving a precise description of its image and of its branch curve (Proposition 1.8). It is worth pointing out that the general surface S contains no irrational pencils (Proposition 1.9).

Section 2 is devoted to the study of the first-order deformations of the surfaces in $\mathcal{M}$ and to the description of the corresponding subset in $\mathcal{M}_{2,2,7}^{\mathrm{can}}$. A key point in our analysis is showing that for all elements in $S \in \mathcal{M}$ we have $h^1(S, T_S) = 3$, see Proposition 2.6.

Since the degree of the Albanese map is in this case a topological invariant (Proposition 2.1), it follows that these surfaces lie in a different connected component of the moduli space than the only other known example with the same invariants, namely the surface with $p_g = q = 2$ and $K^2 = 7$ constructed in [**Ri15**], whose Albanese map is a generically finite *double* cover of an abelian surface with polarization of type $(1, 2)$, see Remark 2.8. Hence the family $\mathcal{M}$ provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$.

Notation and conventions. We work over the field $\mathbb{C}$ of complex numbers. By *surface* we mean a projective, non-singular surface S, and for such a surface K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler-Poincaré characteristic*.

If C is a smooth curve, we identify $\mathrm{Pic}^0(C)$ with the Jacobian variety $J(C)$ by means of the canonical isomorphism provided by the Abel-Jacobi map, see [**BL94**, Theorem 11.1.3]. Furthermore, we write $\mathrm{Sym}^n(C)$ for the n-th symmetric product of C.

Given a finite group G acting on a vector space V, we denote by V^G the G-invariant subspace.

1 The construction

Let V_2 and V_3 be the two hypersurfaces of $\mathbb{P}^3$ defined by

$$V_2 := \{x_2 x_3 + r(x_0, x_1) = 0\}, \quad V_3 := \{x_2^3 + x_3^3 + s(x_0, x_1) = 0\}, \tag{1}$$

where $r, s \in \mathbb{C}[x_0, x_1]$ are general homogeneous forms of degree 2 and 3, respectively. Then $C_4 := V_2 \cap V_3$ is a smooth, canonical curve of genus 4. Denoting by ξ a primitive third root of unity, we see that C_4 admits a free action of the cyclic group $\langle \xi \rangle \cong \mathbb{Z}/3\mathbb{Z}$, defined by

$$\xi \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : x_1 : \xi x_2 : \xi^2 x_3] \tag{2}$$

and the quotient $C_2 := C_4/\langle \xi \rangle$ is a smooth curve of genus 2.

Proposition 1.1. *All étale Galois triple covers of a smooth curve of genus 2 can be obtained in this way.*

Proof. Let C_2 be any smooth curve of genus 2 and choose any étale $\mathbb{Z}/3\mathbb{Z}$-cover $C_4 \longrightarrow C_2$. Thus C_4 is a smooth curve of genus 4 and we can choose a fixed-point free automorphism $c \colon C_4 \longrightarrow C_4$ generating the Galois group of the cover.

The curve C_4 cannot be hyperelliptic, otherwise its 10 Weierstrass points would be an invariant set by any automorphism, which is impossible because any orbit of c consists of three distinct points. Hence the canonical divisor K_{C_4} is very ample and defines an embedding of C_4 in $\mathbb{P}^3 = \mathbb{P}H^0(C_4, K_{C_4})$, whose image (that we still denote by C_4) is the complete intersection of a (uniquely determined) quadric hypersurface V_2 and a cubic hypersurface V_3. It remains to show that we can choose V_2 and V_3 as in (1).

Since the automorphism $c \colon C_4 \longrightarrow C_4$ has no fixed points, Lefschetz holomorphic trace formula ([**GH78**, p. 423]) yields

$$\mathrm{trace}(c^* \,|\, H^0(C_4, K_{C_4})) - \mathrm{trace}(c^* \,|\, H^1(C_4, K_{C_4})) = 0,$$

that is

$$\mathrm{trace}(c^* \,|\, H^0(C_4, K_{C_4})) = 1.$$

This implies that the $\mathbb{Z}/3\mathbb{Z}$-invariant subspace of $H^0(C_4, K_{C_4})$ has dimension 2, whereas the subspaces corresponding to the two non-trivial characters are both 1-dimensional. Then we can choose homogeneous coordinates $[x_0 : x_1 : x_2 : x_3]$ in $\mathbb{P}^3$ such that the action of $\mathbb{Z}/3\mathbb{Z} = \langle \xi \rangle$ is as in (2).

We start by looking at the invariant quadrics in the homogeneous ideal of C_4. There are four invariant monomials of degree 2, namely

$$x_0^2, \ x_0 x_1, \ x_1^2, \ x_2 x_3, \tag{3}$$

hence the invariant subspace $(\mathrm{Sym}^2 H^0(C_4, K_{C_4}))^{\langle \xi \rangle}$ of $\mathrm{Sym}^2 H^0(C_4, K_{C_4})$ has dimension 4. On the other hand, the subspace of invariant quadrics in the homogeneous ideal of C_4 is the kernel of the surjective map

$$(\mathrm{Sym}^2 H^0(C_4, K_{C_4}))^{\langle \xi \rangle} \longrightarrow H^0(C_4, 2K_{C_4})^{\langle \xi \rangle} \cong H^0(C_2, 2K_{C_2}) \cong \mathbb{C}^3,$$

hence it has dimension 1. In other words, the unique quadric V_2 containing C_4 is invariant, hence the polynomial defining V_2 is a linear combination of the monomials in (3). The coefficient of $x_2 x_3$ cannot vanish, or V_2 would be reducible, so V_2 is as in (1).

Let us look now at the invariant cubics in the homogeneous ideal of C_4. There are eight invariant monomials of degree 3, namely

$$x_0^3, \ x_0^2 x_1, \ x_0 x_1^2, \ x_1^3, \ x_0 x_2 x_3, \ x_1 x_2 x_3, \ x_2^3, \ x_3^3,$$

hence the invariant subspace $(\mathrm{Sym}^3 H^0(C_4, K_{C_4}))^{\langle \xi \rangle}$ of $\mathrm{Sym}^3 H^0(C_4, K_{C_4})$ has dimension 8. On the other hand, the subspace of invariant cubics in the homogeneous ideal of C_4 is the kernel of the surjective map

$$(\mathrm{Sym}^3 H^0(C_4, K_{C_4}))^{\langle \xi \rangle} \longrightarrow H^0(C_4, 3K_{C_4})^{\langle \xi \rangle} \cong H^0(C_2, 3K_{C_2}) \cong \mathbb{C}^5,$$

hence it has dimension 3. In particular, this implies that the general invariant cubic hypersurface V_3 containing C_4 is not a multiple of the quadric V_2. Adding suitable scalar multiples of $x_0 V_2$ and $x_1 V_2$ in order to get rid of the monomials $x_0 x_2 x_3$ and $x_1 x_2 x_3$, and changing coordinates by multiplying x_2 and x_3 by suitable constants we obtain an equation of V_3 as in (1) and we are done. $\qquad\square$

Let us consider now the product $C_4 \times C_4 \subset \mathbb{P}^3 \times \mathbb{P}^3$, and write $\mathbf{x} = [x_0 : x_1 : x_2 : x_3]$ for the homogeneous coordinates in the first factor and $\mathbf{y} = [y_0 : y_1 : y_2 : y_3]$ for those in the second factor. Then the action of $\langle \xi \rangle$ on C_4 induces an action of $H := \langle \xi_x, \xi_y, \sigma \rangle$ on $C_4 \times C_4$, where

$$\xi_x(\mathbf{x},\, \mathbf{y}) := (\xi \cdot \mathbf{x},\, \mathbf{y}), \quad \xi_y(\mathbf{x},\, \mathbf{y}) := (\mathbf{x},\, \xi \cdot \mathbf{y}), \quad \sigma(\mathbf{x},\, \mathbf{y}) := (\mathbf{y},\, \mathbf{x}).$$

Clearly ξ_x and ξ_y commute, whereas $\sigma \xi_x = \xi_y \sigma$ and $\sigma \xi_y = \xi_x \sigma$, so H is a semi-direct product of the form

$$H = \langle \xi_x,\, \xi_y \rangle \rtimes \langle \sigma \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}.$$

In particular, $|H| = 18$ and every element $h \in H$ can be written in a unique way as $h = \sigma^k \xi_x^i \xi_y^j$, where $k \in \{0,\, 1\}$ and $i,\, j \in \{0,\, 1,\, 2\}$.

Lemma 1.2. *The non-trivial elements of H having fixed points on $C_4 \times C_4$ are precisely the three elements of order 2*

$$h_i := \sigma \xi_x^i \xi_y^{3-i}, \quad i = 0,\, 1,\, 2.$$

More precisely, the element h_i fixes pointwise the smooth curve

$$\Gamma_i := \{(\mathbf{x},\, \xi^i \cdot \mathbf{x}) \,|\, \mathbf{x} \in C_4\},$$

that is, the graph of the automorphism of C_4 defined by $\mathbf{x} \mapsto \xi^i \cdot \mathbf{x}$. The three curves Γ_0, Γ_1 and Γ_2 are isomorphic to C_4, pairwise disjoint and their self-intersection equals -6.

Proof. Let $h = \sigma^k \xi_x^i \xi_y^j$ be an element of H. If $k = 0$ then $h(\mathbf{x},\, \mathbf{y}) = (\xi^i \cdot \mathbf{x},\, \xi^j \cdot \mathbf{y})$ so, since the action of ξ on C_4 is free, h has fixed points if and only if it is trivial. Thus we can assume $k = 1$, in which case we have

$$\sigma \xi_x^i \xi_y^j(\mathbf{x},\, \mathbf{y}) = (\xi^j \cdot \mathbf{y},\, \xi^i \cdot \mathbf{x}).$$

Hence $(\mathbf{x},\, \mathbf{y})$ is a fixed point for h if and only if $i + j \equiv 0 \,(\mathrm{mod}\ 3)$ and $\mathbf{y} = \xi^i \cdot \mathbf{x}$, that is $(\mathbf{x},\, \mathbf{y}) \in \Gamma_i$.

A straightforward computation using the relations $\sigma^2 = 1$ and $\xi_x \sigma = \sigma \xi_y$ shows that the order of h_i is 2.

The curve Γ_0 is the diagonal of $C_4 \times C_4$, hence it is isomorphic to C_4 and satisfies $(\Gamma_0)^2 = 2 - 2g(C_4) = -6$. The same is true for the curves Γ_1 and Γ_2, because they are the translate of Γ_0 by the action of ξ_y and ξ_x, respectively. Finally, Γ_i and Γ_j are disjoint for $i \neq j$, because ξ acts freely on C_4. $\qquad\square$

Lemma 1.2 implies that the quotient map $C_4 \times C_4 \longrightarrow (C_4 \times C_4)/H$ is ramified exactly over the three curves Γ_i, with ramification index 2 on each of them. We factor such a map through the quotient by the normal abelian subgroup $\langle \xi_x, \xi_y \rangle \cong (\mathbb{Z}/3\mathbb{Z})^2$. This subgroup acts separately on the two factors, whereas σ exchanges them, so we get

$$(C_4 \times C_4)/\langle \xi_x,\, \xi_y \rangle \cong C_2 \times C_2, \quad (C_4 \times C_4)/H \cong \mathrm{Sym}^2(C_2).$$

Thus the surface $B = (C_4 \times C_4)/H$ contains a unique rational curve, namely the (-1)-curve E corresponding to the unique g_2^1 of C_2. Denoting by $\pi\colon B \longrightarrow A$ the blow-down of E, we see that A is an abelian surface isomorphic to the Jacobian variety $J(C_2)$.

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Remark 1.3. Because of Proposition 1.1, all Jacobians of smooth curves of genus 2 can be obtained in this way.

Let us denote now by ξ_{xy} the element $\xi_x \xi_y$ and set $G := \langle \xi_{xy}, \sigma \rangle$; then G is a non-normal, abelian subgroup of H, isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Setting

$$T := (C_4 \times C_4)/\langle \xi_{xy} \rangle, \quad S := (C_4 \times C_4)/G,$$

and writing $t\colon C_4 \times C_4 \longrightarrow T$ and $f\colon C_4 \times C_4 \longrightarrow S$ for the corresponding projection maps, we have the following commutative diagram:

$$\text{(4)}$$

Note that $u\colon T \longrightarrow S$ is a double cover, induced by the involution σ exchanging the two coordinates in $C_4 \times C_4$.

We first compute the invariants of T.

Lemma 1.4. *The surface T is minimal of general type with*

$$p_g(T) = 6, \quad q(T) = 4, \quad K_T^2 = 24.$$

Proof. By standard calculations we have

$$p_g(C_4 \times C_4) = 16, \quad q(C_4 \times C_4) = 8, \quad K_{C_4 \times C_4}^2 = 72.$$

The group $\langle \xi_{xy} \rangle \cong \mathbb{Z}/3\mathbb{Z}$ acts diagonally and freely on $C_4 \times C_4$, hence T is a so-called *quasi-bundle*, see for instance [**Pol09**, Section 3]. Therefore we obtain

$$K_T^2 = \frac{1}{3} K_{C_4 \times C_4}^2 = 24, \quad \chi(\mathcal{O}_T) = \frac{1}{3} \chi(\mathcal{O}_{C_4 \times C_4}) = 3, \quad q(T) = g(C_2) + g(C_2) = 4,$$

so $p_g(T) = 6$. Note that by Noether's formula this implies $c_2(T) = 12$. Finally, T is a minimal surface of general type because it a finite, étale quotient of the minimal surface of general type $C_4 \times C_4$. $\qquad \square$

The three curves $\Gamma_i \subset C_4 \times C_4$ are ξ_{xy}-invariant, hence their images $\Sigma_i := t(\Gamma_i) \subset T$ are three curves isomorphic to C_2 and such that $(\Sigma_i)^2 = \frac{1}{3}(\Gamma_i)^2 = -2$. Moreover, the curve Γ_0 is also σ-invariant, whereas Γ_1 and Γ_2 are switched by the action of σ. Then $D_S := u(\Sigma_0)$ and $R := u(\Sigma_1) = u(\Sigma_2)$ are two disjoint curves in S, both isomorphic to C_2, such that $(D_S)^2 = -4$ and $R^2 = -2$. Note that D_S is the branch locus of the double cover $u\colon T \longrightarrow S$.

We can now compute the invariants of S.

Proposition 1.5. *The surface S is a minimal surface of general type with*

$$p_g(S) = 2, \quad q(S) = 2, \quad K_S^2 = 7.$$

The morphism $\beta \colon S \longrightarrow B$ is a non-Galois triple cover, simply ramified over R and simply branched over the diagonal $D_B \subset B$. Finally, S contains no rational curves (in particular, K_S is ample) and contains a smooth elliptic curve, namely $Z := \beta^* E$ (which satisfies $Z^2 = -3$).

Proof. We start by proving the last claim. The two smooth curves D_B and E intersect transversally at the six points corresponding to the six Weierstrass points of C_2, so $Z := \beta^* E$ is a smooth, irreducible curve of genus 1 contained in S, such that

$$ZR = (\beta^* E).R = E.(\beta_* R) = ED_B = 6. \tag{5}$$

On the other hand, S does not contain any rational curve. Otherwise, such a curve would map would map onto E via $\beta \colon S \longrightarrow B$, impossible because we have seen that $\beta^* E$ is smooth of genus 1.

Since the double cover $u \colon T \longrightarrow S$ is branched over the curve D_S, it follows that D_S is 2-divisible in $\mathrm{Pic}(S)$ and moreover

$$24 = K_T^2 = 2\left(K_S + \frac{1}{2} D_S\right)^2.$$

Using $(D_S)^2 = -4$ and $K_S D_S = 6$, we find $K_S^2 = 7$. Since S does not contain any rational curve and $K_S^2 > 0$, we deduce that S is a minimal surface of general type with ample canonical class.

Now, as $K_B = \pi^* K_A + E = E$, the Riemann-Hurwitz formula yields

$$K_S = \beta^* K_B + R = Z + R, \tag{6}$$

and this allows us to compute Z^2. In fact, using (5) and (6), we can write

$$7 = K_S^2 = Z^2 + 2ZR + R^2 = Z^2 + 10,$$

that is $Z^2 = -3$.

Next, denoting by χ_{top} the topological Euler number, we have

$$\chi_{\mathrm{top}}(S - D_S - R) = \frac{1}{2} \chi_{\mathrm{top}}(T - \Sigma_0 - \Sigma_1 - \Sigma_2)$$

$$= \frac{1}{2}(c_2(T) - \chi_{\mathrm{top}}(\Sigma_0) - \chi_{\mathrm{top}}(\Sigma_1) - \chi_{\mathrm{top}}(\Sigma_2)) = \frac{1}{2}(12 - 3(-2)) = 9,$$

so

$$c_2(S) = \chi_{\mathrm{top}}(S) = \chi_{\mathrm{top}}(S - D_S - R) + \chi_{\mathrm{top}}(D_S) + \chi_{\mathrm{top}}(R) = 9 - 2 - 2 = 5.$$

Therefore Noether's formula yields $\chi(\mathcal{O}_S) = 1$, that is $p_g(S) = q(S)$.

The existence of the surjective morphism $\alpha \colon S \longrightarrow A$ implies $q \geq 2$, and since minimal surfaces of general type with $p_g = q \geq 3$ have either $K^2 = 6$ or $K^2 = 8$ (see for instance [**BaCaPi06**]), we deduce $p_g(S) = q(S) = 2$.

The morphism $\beta \colon S \longrightarrow B$ is a non-Galois triple cover, because G is a non-normal subgroup of index 3 in H. Since $t \colon C_4 \times C_4 \longrightarrow T$ is étale and $u \colon T \longrightarrow S$ is branched over D_S, by Lemma 1.2 it follows that $\beta \colon S \longrightarrow B$ is simply ramified over R, and hence simply branched over $\beta(R) = D_B$. $\qquad \square$

Remark 1.6. It is also possible to directly compute $q(S)$ without exploiting the classification of surfaces with $p_g = q \geq 3$, for instance by using [**CanFr15**, Proposition 2.6].

Let us now identify the blow-up morphism $\pi \colon B \longrightarrow A$ with the Abel-Jacobi map

$$\mathrm{Sym}^2(C_2) \longrightarrow J(C_2).$$

If Θ is the class of a theta divisor in $\mathrm{NS}(A)$, let us define the class $\Theta_B := \pi^*\Theta$ in $\mathrm{NS}(B)$. Moreover, let us write x for the class in $\mathrm{NS}(B)$ given by the image of the map

$$C_2 \longrightarrow \mathrm{Sym}^2(C_2), \quad p \mapsto p_0 + p$$

where $p_0 \in C_2$ is fixed (such a class does not depend on p_0). Then we can prove the following

Lemma 1.7. *The equality* $\pi_* D_B = 4\Theta$ *holds in* $\mathrm{NS}(A)$.

Proof. This is a consequence of more general results on g-fold symmetric products of curves of genus g, see for instance [**Mat62**, equations (1) and (5)]. For $g = 2$, in our notations such equations become the following relations in $\mathrm{NS}(B)$:

$$2E + D_B = 4x, \quad \Theta_B = E + x,$$

that is $D_B = 4\Theta_B - 6E$. Then we get the result by applying the push-forward map $\pi_* \colon \mathrm{NS}(B) \longrightarrow \mathrm{NS}(A)$. $\qquad\square$

The next step consists in describing the Albanese morphism of S.

Proposition 1.8. *The generically finite triple cover* $\alpha = \pi \circ \beta \colon S \longrightarrow A$ *is the Albanese map of* S, *and the only curve contracted by it is* Z. *The morphism* α *is branched over a divisor* D_A *numerically equivalent to* 4Θ, *having an ordinary sextuple point and no other singularities.*

Proof. By the universal property of the Albanese variety ([**Be96**, Chapter V]), the morphism $\alpha \colon S \longrightarrow A$ must factor through the Albanese morphism of S; but α is surjective and generically of degree 3, so it must actually coincide with the Albanese morphism of S. Since β is a finite morphism, α only contracts the preimage of E in S, which is Z. The branch locus D_A of α is equal to the image of the diagonal D_B via $\pi \colon B \longrightarrow A$; since D_B is smooth and intersects E transversally at six points, it follows that D_A has an ordinary sextuple point and no other singularities. Finally, the fact that D_A is numerically equivalent to 4Θ follows from Lemma 1.7. $\qquad\square$
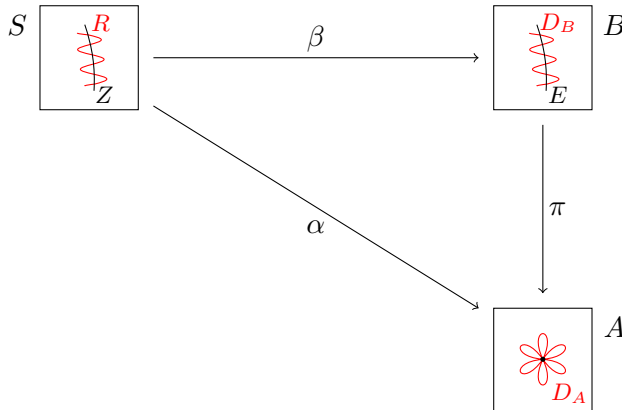
The situation is summarized in Figure 1 below.

Figure 1: The triple covers α and β

Furthermore, the Stein factorization of $\alpha\colon S \longrightarrow A$ is described in the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\ c_Z\ } & \tilde{S} \\
& {\scriptstyle \alpha}\searrow & \downarrow{\scriptstyle \tilde{\alpha}} \\
& & A,
\end{array}
$$

where $c_Z\colon S \longrightarrow \tilde{S}$ is the birational morphism given by the contraction of the elliptic curve Z. Since $Z^2 = -3$, the normal surface $\tilde{S}$ has a Gorenstein elliptic singularity of type $\tilde{E}_6$, see [**Is10**, Theorem 7.6.4].

Recall that an *irrational pencil* (or *irrational fibration*) on a smooth, projective surface is a surjective morphism with connected fibres over a curve of positive genus.

Proposition 1.9. *The general surface S contains no irrational pencils.*

Proof. Assume that $\phi\colon S \longrightarrow W$ is an irrational pencil on S. Since $q(S) = 2$, we have either $g(W) = 1$ or $g(W) = 2$. On the other hand, using the embedding $W \hookrightarrow J(W)$ and the universal property of the Albanese map, we obtain a morphism $A \longrightarrow J(W)$ whose image is isomorphic to the curve W. This rules out the case $g(W) = 2$, hence W is an elliptic curve and so A is a non-simple abelian surface. The proof is now complete, because A is isomorphic to the Jacobian variety $J(C_2)$, which is well-known to be simple for a general choice of C_2. $\qquad\square$

2   The moduli space

A projective variety X is called *of maximal Albanese dimension* if its Albanese map $\alpha_X\colon X \longrightarrow \mathrm{Alb}(X)$ is generically finite onto its image. For surfaces of general type with irregularity at least 2, this is actually a topological property, as shown by the result below.

Proposition 2.1. *Let S be a minimal surface of general type with $q(S) \geq 2$. If S is of maximal Albanese dimension, then the same holds for any surface which is homeomorphic to S. Furthermore, in the case $q(S) = 2$ the degree of the Albanese map $\alpha\colon S \longrightarrow A$ is a topological invariant.*

Proof. This follows by the results of [**Ca91**], see for instance [**PePol13b**, Proposition 3.1]. $\qquad\square$

Proposition 2.1 allows us to study the deformations of S by relating them to those of the flat triple cover $\beta\colon S \longrightarrow B$. Since the trace map provides a splitting of the short exact sequence

$$
0 \longrightarrow \mathcal{O}_B \longrightarrow \beta_*\mathcal{O}_S \longrightarrow \mathcal{E}_\beta \longrightarrow 0,
$$

we obtain a direct sum decomposition

$$
\beta_*\mathcal{O}_S = \mathcal{O}_B \oplus \mathcal{E}_\beta, \tag{7}
$$

where $\mathcal{E}_\beta$ is a vector bundle of rank 2 on B which satisfies

$$
h^0(B, \mathcal{E}_\beta) = 0, \quad h^1(B, \mathcal{E}_\beta) = 0, \quad h^2(B, \mathcal{E}_\beta) = 1. \tag{8}
$$

According to [**Mir85**], we call $\mathcal{E}_\beta$ the *Tschirnhausen bundle* of β. By [**Sern06**, p. 162] we have an exact sequence

$$
0 \longrightarrow T_S \xrightarrow{\ d\beta\ } \beta^* T_B \longrightarrow \mathcal{N}_\beta \longrightarrow 0, \tag{9}
$$

where $\mathcal{N}_\beta$ is a coherent sheaf supported on R called the *normal sheaf of* β. More precisely, since R is smooth, by [**Rol10**, Lemma 3.2] we obtain

$$\mathcal{N}_\beta = (N_{R/S})^{\otimes 2} = \mathcal{O}_R(2R). \tag{10}$$

There is also another exact sequence

$$0 \longrightarrow T_B \xrightarrow{d\pi} \pi^* T_A \longrightarrow \mathcal{O}_E(-E) \longrightarrow 0, \tag{11}$$

see [**Sern06**, p. 73]. Passing to cohomology in (11), a standard computation as in [**Ha10**, Exercise 10.5 p. 83] yields

$$h^0(B, T_B) = 0, \quad h^1(B, T_B) = 4, \quad h^2(B, T_B) = 2. \tag{12}$$

Lemma 2.2. *We have $h^0(S, \beta^* T_B) = 0$. Hence there are no first-order deformations of the morphism $\beta \colon S \longrightarrow B$ keeping both S and B fixed.*

Proof. Since β is a finite morphism, using the projection formula together with (7) we infer

$$h^i(S, \beta^* T_B) = h^i(B, \beta_* \beta^* T_B) = h^i(B, T_B) + h^i(B, T_B \otimes \mathcal{E}_\beta)$$

for all $i \in \{0, 1, 2\}$, so by (12) it is sufficient to show that $h^0(B, T_B \otimes \mathcal{E}_\beta) = 0$. Tensoring (11) with $\mathcal{E}_\beta$, passing to cohomology and using the fact that T_A is a trivial bundle of rank 2, we obtain

$$H^0(B, T_B \otimes \mathcal{E}_\beta) \subseteq H^0(B, (\pi^* T_A) \otimes \mathcal{E}_\beta) = H^0(B, \mathcal{E}_\beta)^{\oplus 2}$$

and the last vector space is zero by (8). The statement about the first-order deformations of β follows by [**Sern06**, Proposition 3.4.2]. $\qquad\square$

As in [**PePol13b**, Section 3] we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & (13) \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \xrightarrow{d\beta} & \beta^* T_B & \longrightarrow & \mathcal{N}_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_S & \xrightarrow{d\alpha} & \alpha^* T_A & \longrightarrow & \mathcal{N}_\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_Z(-Z) & \longrightarrow & \mathcal{O}_Z(-Z) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

whose central column is the pullback of (11) via $\beta \colon S \longrightarrow B$. Note that the normal sheaf $\mathcal{N}_\alpha$ of $\alpha \colon S \longrightarrow A$ is supported on the set of critical points of α, namely on the reducible divisor $R + Z$.

Lemma 2.3. *We have*

$$h^1(S, T_S) = h^0(R + Z, \mathcal{N}_\alpha) + 1 \geq 3.$$

Proof. Let us write down the cohomology exact sequence associated with the short exact sequence in the central row of (13), recalling first that S is a surface of general type and therefore $h^0(S, T_S) = 0$:

$$0 \longrightarrow H^0(S,\, \alpha^*T_A) \cong \mathbb{C}^2 \longrightarrow H^0(R+Z,\, \mathcal{N}_\alpha) \longrightarrow H^1(S,\, T_S) \xrightarrow{\;\varepsilon\;} H^1(S,\, \alpha^*T_A) \cong H^1(A,\, T_A)$$

Then the claim will follow if we show that $\mathrm{rank}(\varepsilon) = 3$, and this can be done by using the same argument as in [**PePol13b**, Section 3]. More precisely, by Remark 1.3 the first-order deformations of S dominate the first-order algebraic deformations of A, so $\mathrm{rank}(\varepsilon) \geq 3$; on the other hand, the Albanese variety of every deformation of S has to remain algebraic, so $\mathrm{rank}(\varepsilon) \leq 3$ and we are done. $\qquad\square$

Thus, in order to understand the first-order deformations of S, we can study $\mathcal{N}_\alpha$.

Lemma 2.4. *The sheaf $\mathcal{N}_\alpha$ is locally free of rank 1 on the reducible curve $R + Z$.*

Proof. By a standard application of Nakayama's lemma (see for instance [**Kempf93**, Corollary 5.3.4]), it suffices to check that the $\mathbb{C}$-vector space $\mathcal{N}_{\alpha,x}/\mathfrak{m}_x \mathcal{N}_{\alpha,x}$ has dimension 1 for all $x \in R + Z$, where $\mathfrak{m}_x \subset \mathcal{O}_{R+Z,\,x}$ is the maximal ideal. Equivalently, we will check that the vector bundle map $d\alpha\colon T_S \longrightarrow \alpha^*T_A$ has rank 1 at each point $x \in R + Z$. Let us distinguish three cases.

- If $x \in R \setminus Z$, then α is locally of the form $(u,\, v) \mapsto (u^2,\, v)$, with $x = (0,\, 0)$ and R given by $u = 0$. Then $d\alpha$ is the linear map associated with the matrix

$$\begin{pmatrix} 2u & 0 \\ 0 & 1 \end{pmatrix},$$

  which has rank 1 at the point x.

- If $x \in Z \setminus R$, then α is locally a smooth blow-up, hence of the form $(u,\, v) \mapsto (uv,\, v)$, where $x = (0,\, 0)$ and Z corresponds to the exceptional divisor, whose equation is $v = 0$. Then $d\alpha$ is the linear map associated with the matrix

$$\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix},$$

  which has rank 1 at the point x.

- Finally, if $x \in R \cap Z$ then α is locally the composition of the two maps above, so of the form $(u,\, v) \mapsto (u^2v,\, v)$, where $x = (0,\, 0)$, the curve R corresponds to the locus $u = 0$ and the curve Z to the locus $v = 0$. Then $d\alpha$ is the linear map associated with the matrix

$$\begin{pmatrix} 2uv & u^2 \\ 0 & 1 \end{pmatrix},$$

  which has rank 1 at the point x.

This completes the proof. $\qquad\square$

We can be more precise and compute the restrictions of $\mathcal{N}_\alpha$ to both curves R and Z.

Lemma 2.5. *We have*

$$\mathcal{N}_{\alpha|Z} = \mathcal{O}_Z(-Z), \quad \mathcal{N}_{\alpha|R} = \mathcal{O}_R(2R + Z) = \mathcal{O}_R(K_R).$$

Proof. Let us first apply the functor $\otimes_{\mathcal{O}_{R+Z}}\mathcal{O}_Z$ to the exact sequence forming the last column of diagram (13); using (10), we get

$$\mathcal{O}_R(2R) \otimes \mathcal{O}_Z \xrightarrow{\zeta} \mathcal{N}_{\alpha|Z} \longrightarrow \mathcal{O}_Z(-Z) \longrightarrow 0.$$

By Lemma 2.4, the sheaf $\mathcal{N}_{\alpha|Z}$ is locally free on Z; on the other hand, $\mathcal{O}_R(2R) \otimes \mathcal{O}_Z$ is a torsion sheaf, hence ζ is the zero map and so $\mathcal{N}_{\alpha|Z} \cong \mathcal{O}_Z(-Z)$.

Next, we apply to the same exact sequence the functor $\otimes_{\mathcal{O}_{R+Z}}\mathcal{O}_R$, obtaining

$$\mathcal{T} \xrightarrow{\tau} \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \tag{14}$$

Since $\mathcal{T} := \mathrm{Tor}^1_{\mathcal{O}_{R+Z}}(\mathcal{O}_Z(-Z), \mathcal{O}_R)$ is a torsion sheaf (supported on $R \cap Z$) and $\mathcal{O}_R(2R)$ is locally free on R, we deduce that τ is the zero map and so (14) becomes

$$0 \longrightarrow \mathcal{O}_R(2R) \longrightarrow \mathcal{N}_{\alpha|R} \longrightarrow \mathcal{O}_Z(-Z) \otimes \mathcal{O}_R \longrightarrow 0. \tag{15}$$

On the other hand, the curves R and Z intersect transversally at the six Weierstrass points $p_1, \ldots p_6$ of R, so we infer

$$\mathcal{O}_Z(-Z) \otimes \mathcal{O}_R = \mathcal{O}_Z \otimes \mathcal{O}_R = \bigoplus_{i=1}^{6} \mathcal{O}_{p_i}. \tag{16}$$

Hence (15) and (16) yield

$$0 \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{N}_{\alpha|R}(-2R) \longrightarrow \bigoplus_{1}^{6} \mathcal{O}_{p_i} \longrightarrow 0,$$

that is the invertible sheaf $\mathcal{N}_{\alpha|R}(-2R)$ has a global section whose divisor is $\sum p_i$. This means $\mathcal{N}_{\alpha|R} \cong \mathcal{O}_R(2R + \sum p_i) = \mathcal{O}_R(2R + Z)$. Finally, equation (6) shows that $R + Z$ is a canonical divisor on S, so by using adjunction formula we obtain

$$\mathcal{O}_R(2R + Z) = \mathcal{O}_S(K_S + R) \otimes \mathcal{O}_R = \mathcal{O}_R(K_R).$$

$\square$

We can finally prove

Proposition 2.6. *All surfaces S constructed in* Section 1 *satisfy*

$$h^1(S, T_S) = 3.$$

Proof. By Lemma 2.3 it suffices to show the inequality $h^0(R+Z, \mathcal{N}_\alpha) \leq 2$. By [**BHPV03**, p. 62] we have a "decomposition sequence"

$$o \longrightarrow \mathcal{O}_Z(-R) \longrightarrow \mathcal{O}_{R+Z} \longrightarrow \mathcal{O}_R \longrightarrow 0,$$

which gives, tensoring with $\mathcal{N}_\alpha$ and using Lemma 2.5,

$$0 \longrightarrow \mathcal{O}_Z(-R-Z) \longrightarrow \mathcal{N}_\alpha \longrightarrow \mathcal{O}_R(K_R) \longrightarrow 0.$$

Since $Z(-R-Z) = -3 < 0$, we deduce $H^0(Z, \mathcal{O}_Z(-R-Z)) = 0$. So $H^0(R+Z, \mathcal{N}_\alpha)$ injects into $H^0(R, K_R) = \mathbb{C}^2$ and we are done. $\square$

The moduli space of principally polarized abelian surfaces has dimension 3 and the curve D_A has only trivial deformations in A (because the diagonal D_B is rigid in B), so our surfaces S provide a 3-dimensional subset $\mathcal{M}$ of the moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,7}$ of (canonical models of) minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$. Because of Proposition 2.6, the corresponding Kuranishi family is smooth; this implies that $\mathcal{M}$ has at most quotient singularities, so it is a normal (and hence generically smooth) open subset of $\mathcal{M}^{\mathrm{can}}_{2,2,7}$. In particular, $\mathcal{M}$ provides a dense open set of a generically smooth, irreducible component of this moduli space.

Summing up, we have proven the Main Theorem stated in the introduction, namely

Theorem 2.7. *There exists a 3-dimensional family $\mathcal{M}$ of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ such that, for all elements $S \in \mathcal{M}$, the canonical class K_S is ample and the Albanese map $\alpha \colon S \longrightarrow A$ is a generically finite triple cover of a principally polarized abelian surface (A, Θ), simply branched over a curve D_A numerically equivalent to 4Θ having an ordinary sextuple point and no other singularities. The family $\mathcal{M}$ provides a generically smooth, irreducible, open and normal subset of the Gieseker moduli space $\mathcal{M}^{\mathrm{can}}_{2,2,7}$ of canonical models of minimal surfaces of general type with $p_g = q = 2$ and $K^2 = 7$.*

Remark 2.8. By Proposition 2.1 the degree of the Albanese map is in our case a topological invariant, so it follows that the surfaces in $\mathcal{M}$ lie in a different connected component of $\mathcal{M}^{\mathrm{can}}_{2,2,7}$ than the only other known example with the same invariants, namely the surface with $p_g = q = 2$ and $K^2 = 7$ constructed in [**Ri15**], whose Albanese map is a generically finite *double* cover of an abelian surface with polarization of type $(1, 2)$. Hence the family $\mathcal{M}$ provides a substantially new piece in the fine classification of minimal surfaces of general type with $p_g = q = 2$.

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